Lesson 6. Nonstationary Poisson Processes

1 Overview

- We've been studying Poisson processes with a **stationary** arrival rate λ
  - In other words, λ doesn't change over time
- This lesson: what happens when the arrival rate is **nonstationary**?
  - In other words, the arrival rate $\lambda(\tau)$ is a function of time τ
- Main idea: we transform a stationary Poisson process with arrival rate 1 into a **nonstationary Poisson process** with a time-dependent arrival rate

2 Integrated rate functions

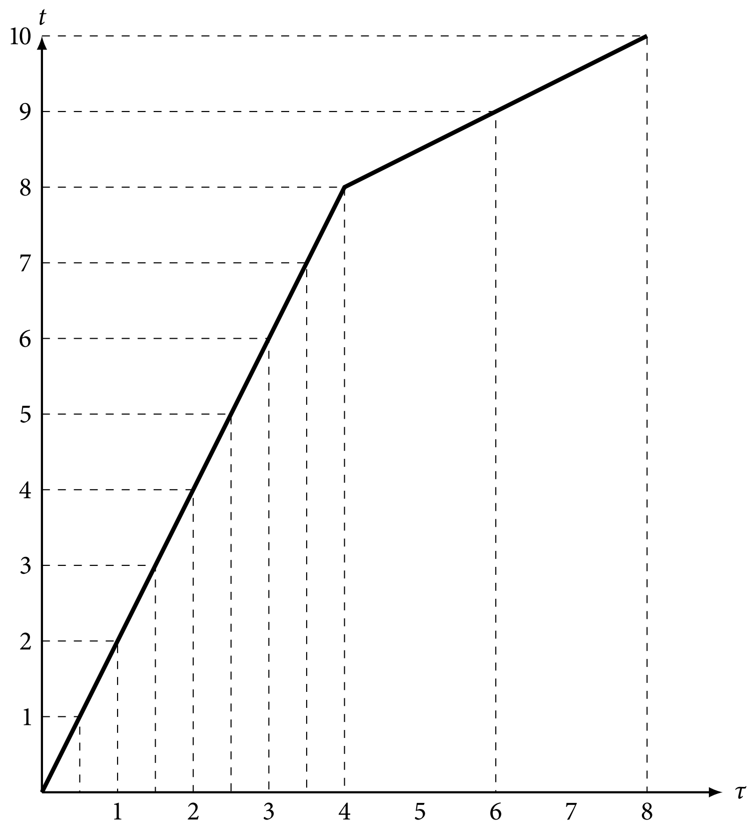
You have been put in charge of studying the operations at a helicopter maintenance facility. The data indicates that the facility is busier in the morning than in the afternoon. In the morning (8:00 - 12:00), the average time between helicopters arrivals is 0.5 hours. On the other hand, in the afternoon (12:00 - 16:00), the average time between helicopter arrivals is 2 hours.

- Let's say that $\tau = 0$ corresponds to 8:00
- Therefore, the (expected) arrival rate $\lambda(\tau)$ as a function of τ (in hours) is:

- We can compute the expected number of arrivals by time τ:

- $\Lambda(\tau)$ is called the **integrated-rate function**
- For the arrival rate $\lambda(\tau)$ given above, the integrated-rate function is

- A graph of the integrated-rate function $\Lambda(\tau)$:



- The inverse of the integrated-rate function $\Lambda(\tau)$:

- Key idea: τ and t represent <u>different time scales</u> connected by $t = \Lambda(\tau)$ or $\tau = \Lambda^{-1}(t)$
  - t represents the time scale for a stationary Poisson process with arrival rate 1
  - τ represents the time scale of a nonstationary Poisson process
- Why does this work? Intuitively, can be seen from the graph above

3 Nonstationary Poisson processes, formally

- Consider a Poisson process with arrival rate 1 with:
  - Y_t = number of arrivals by time t
  - T_n = time of nth arrival
- We can transform this into a **nonstationary Poisson process** with integrated-rate function $\Lambda(\tau)$:
  - Z_τ = ______ = number of arrivals by time τ
  - U_n = ______ = time of nth arrival
- The number of arrivals in the interval $(\tau, \tau + \Delta\tau]$ is
- Therefore, $E[Z_{\tau+\Delta\tau} - Z_\tau] =$ ______
- A nonstationary Poisson process satisfies the independent-increments property:
- The probability distribution of the number of arrivals in $(\tau, \tau + \Delta\tau]$ depends on both $\Delta\tau$ and τ

  $\Rightarrow$ The stationary-increments and memoryless properties no longer apply
- Proofs on page 112 of Nelson

Example 1. In the maintenance facility example above:

a. What is the probability that 7 helicopters arrive between 8:00 and 13:00, given that 5 arrived between 8:00 and 11:00?
b. What is the expected number of helicopters to arrive between 10:00 and 14:00?

Example 2. Cantor's Car Repair is open from 9:00 ($\tau = 0$) to 15:00 ($\tau = 360$). Customers arrive according to a nonstationary Poisson process; the arrival rate at time τ is

$$\lambda(\tau) = \begin{cases} 1/6 & \text{if } 0 \leq \tau < 180, \\ 1/5 & \text{if } 180 \leq \tau < 360 \end{cases}$$

a. Find the integrated rate function $\Lambda(\tau)$. What does $\Lambda(\tau)$ mean in the context of the problem?
b. What is the probability that 5 customers arrive between 11:00 and 13:00?
c. What is the expected number of customers that arrive between 11:00 and 13:00?
d. If 15 customers have arrived by 11:00, what is the probability that more than 60 customers will have arrived throughout the course of the day?

4 Exercises

Problem 1. The Simplexville Emergency Dispatch receives phone calls according to a nonstationary Poisson arrival process with integrated rate function

$$\Lambda(\tau) = \begin{cases} 3\tau & \text{if } 0 \leq t < 8 \\ 5\tau - 16 & \text{if } 8 \leq t < 20 \\ \frac{3}{2}\tau + 54 & \text{if } 20 \leq t \leq 24 \end{cases}$$

where τ is in hours and $\tau = 0$ corresponds to 0:00.

a. What is the probability that 12 or fewer phone calls have been received between 18:00 and 22:00?
b. If exactly 40 phone calls have been received between 0:00 and 12:00, what is the probability that 80 or more phone calls have been received over the course of the entire day (0:00 - 24:00)?
c. In words, briefly describe the meaning of $\Lambda(24)$ in the context of this problem.